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Dynamics of Intelligent Systems

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Recent advances in nonlinear dynamics demonstrate a remarkable complexity of patterns outside of equilibrium, which are derived from simple basic laws of physics. A class of mathematical models has been identified providing a variety of such patterns in the form of static, periodic, or chaotic attractors. These models appear to be so general that they predict not only physical, but also biological, economic, and social patterns of behavior. Such a phenomenological reductionism may suggest that, on the dynamical level of description, there is no difference between a solar system, a swarm of insects, and a stock market. However, this conclusion is wrong for a very simple reason: Even primitive living species possess additional non-Newtonian properties which are not included in the laws of Newtonian or statistical mechanics. These properties follow from a privileged ability of living species to possess a self-image (a concept introduced in mathematical psychology). In this paper we consider the existence of a self-image as a postulate to be added to classical physics for modeling behavior of living systems. We show that self-image can be incorporated into the mathematical formalism of a nonlinear dynamics which evolves in probability space. We demonstrate that one of the basic invariants of living systems is their ability to predict the future, which is associated with intelligence.

1. INTRODUCTION

Modeling of life can be performed on many different levels of description. In this paper we will be concerned with geometrical invariants of biosignatures representing prints of behavioral patterns. One of the most remarkable patterns in biology is the formation of species aggregation as an evolutionarily advantageous state in which members derive benefits of protection, mate choice, and centralized information, balanced by the costs of limiting resources. Consisting of individual members, aggregations nevertheless function as an integrated whole, displaying a complex set of behaviors

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not possible at the level of the individual organism. Aggregation occurs at all sizes from bacteria to whales, from groups of 10 to 10 million. Therefore, in the world of bacteria, biosignatures should be associated with configurations of aggregations. Operationally, aggregations fit into two classes: Those that self-organize and those that form in response to external objectives (light, food). They may have a variety of geometrical forms (milling, in which individual members circle about an unoccupied core, polarity without a leader, a distinct shape whose topology varies to suit the tasks, etc.). Each such configuration can be associated with a story which explains its evolutionary advantage [2].

The most powerful modeling tool for analysis of biological patterns is based upon the fundamental concept of nonlinear dynamics called an attractor. An attractor is a stable dissipative structure which does not depend (at least, within a certain basin) upon the initial conditions. Due to this property, the whole history of evolution prior to the attraction becomes irrelevant, which represents a great advantage for information processing, and in particular for pattern recognition.

The mathematical approach to pattern formation is based upon the theory of active systems (both natural and artificial) which is described by a system of PDE [1]

$$\dot{a}_i = g_i(\{a\}) + \sum_j \beta_{ij}(\nabla a_j)^2 + \sum_{ij} D_{ij}\nabla^2 a_j, \qquad 1, 2, \ldots, n \tag{1}$$

where $\{a\} = a_1 \ldots a_n$ are state variables, g is a multiextremal function, and β_{ij}, D_{ij} are constants.

The properties of the solutions to Eq. (1) in terms of the type of attractor depend upon a certain dimensionless control parameter R (such as the Reynolds or the Rayleigh number). This dependence may lose its uniqueness at certain critical points when $R = R_{cr}$, and the solution becomes linearly unstable. Because of the richness of postinstability structures, many different stable patterns (both deterministic or chaotic) may appear when $R = R_{cr}$. These structures include effects of fluid dynamics, nonlinear diffusion, chemical kinetics, etc., and its solutions can form such patterns as Bernard cells, Taylor vortices, trigger and spiral waves, traveling pulses, etc. Surprisingly, the same equations are exploited for simulating biological patterns such as transplantation and regeneration in hydra, compartment formation in *Drosophila*, mammalian coat markings, pigment patterns on mollusk shells, etc. However, such a “universality” of Eq. (1) immediately disqualifies it as a tool for the detection of life since it does not suggest any mechanisms for life–nonlife discrimination. Therefore it is not a coincidence that the main success in simulating biological patterns using Eq. (1) is associated with morphogenesis, i.e., with structures during the growth of an organism rather than with a collective behavior of swarms or colonies. Indeed, in contrast to a set of physical particles, which interact via flows of energy, living species interact via flows of information, which is not captured by Eq. (1). The flow of information is produced and processed by a signaling system whose complexity ranges from interactions between single molecules to interactions between species in ecological systems, and may include receptors, transducers, enzymes, diffusible second messengers, etc. [2]. In order to incorporate signaling phenomena into the process of pattern formation on the same level of description, a phenomenological approach based upon ideas proposed in ref. 3 will be developed in the next section.

2. REFLEXIVE DYNAMICS

In contrast to physical systems, from the viewpoint of nonlinear dynamics a biological system can be considered as a multibody system (with “bodies” represented by cells) interconnected via information flows. Since these flows as well as responses to them may be distorted, delayed, or incomplete, the motion of each cell becomes stochastic, and it can be simulated by a controlled random walk. This random walk is caused not by an external noise (as in the case of a physical particle), but by an internal effort (a “free will”) triggered by the signaling system. Physically it is represented by an ordered sequence of runs, pauses, and tumbles.

One of the main challenges in modeling living systems is to distinguish a random walk of physical origin (for instance, Brownian motion) from one of biological origin; this will constitute the starting point of the proposed approach. As conjectured in ref. 3, a biological random walk must be nonlinear. Indeed, any stochastic Markov process can be described by a linear Fokker–Planck equation (or its discretized version); only those types of processes have been observed in the inanimate world. However, all such processes always converge to a stable (ergodic or periodic) state, i.e., to states of lower complexity and higher entropy. At the same time, the evolution of living systems is directed toward a higher level of complexity if complexity is associated with a number of structural variations. The simplest way to mimic such a tendency is to incorporate a nonlinearity into the random walk; then the probability evolution will attain the features of the Burgers equation: the formation and dissipation of shock waves initiated by small shallow wave disturbances. As a result, the evolution never “dies”: it produces new, different configurations which are accompanied by increase or decrease of entropy (the decrease takes place during formation of shock waves, the increase during their dissipation). In other words, the evolution can be directed “against the second law of thermodynamics” by forming patterns outside of equilibrium.

In order to elucidate both the physical and the biological aspects of the proposed model, let us start with a one-dimensional random walk:

$$x_{t+\tau} = x_t + h \operatorname{sgn} [R + \mu], \qquad h = \text{const}, \qquad \tau = \text{const} \tag{2}$$

where h and τ are the space (along x) and time steps, respectively, R is a random function taking values from -1 to 1 with equal probability, and μ is a control parameter ($|\mu| \leq 1/2$). At a reflecting boundary $\mu = \pm 1/2$. (Physical implementations of this model are discussed in the Appendix).

Equation (1) describes motion in actual physical space. But since this motion is irregular, it is more convenient to turn to probability space:

$$f_{t+\tau,x} = pf_{t,x-h} + (1-p)\, f_{t,x+h}; \qquad f_{t+\tau} = f(t+\tau, x) \tag{3}$$

where $f(x, t)$ is the probability that the moving particle occupies the point x at the instant t, and the transition probability

$$p = \frac{1}{2} + \mu, \qquad 0 \leq p \leq 1 \tag{4}$$

If the system interacts with the external world, i.e.,

$$\mu = \mu(x) \qquad \text{and therefore} \qquad p = p(x)$$

then the solution to Eq. (3) subject to the reflecting boundary conditions converges to a stable stochastic attractor [4, 5]. However, if

$$\mu = \mu(f) \qquad \text{and therefore} \quad p = p(f) \tag{5}$$

Eq. (3) becomes nonlinear, and Eq. (2) is coupled with Eq. (3) via the feedback (4).

From the physical viewpoint, the system (2), (3) can be compared with the Langevin equation, which is coupled with the corresponding Fokker–Planck equation such that the stochastic force is fully defined by the current probability distributions, while the diffusion coefficient is fully defined by the stochastic force. The process described by this system is Markovian since the future still depends only upon the present, and not the past. However, now the present includes not only values of the state variable, but also its probability distribution, and that leads to the nonlinear evolution of the random walk.

From the mathematical viewpoint, Eq. (2) simulates probabilities and Eq. (3) manipulates their values. The connection between these equations is the following: if Eq. (2) is run independently many times and a statistical analysis of these solutions is performed, then the calculated probability will evolve according to Eq. (3).

From the biological viewpoint, Eqs. (2) and (3) represent the same subject: the simplest living species, or, using terminology introduced by

Leibniz, a monad. Equation (2) simulates its motor dynamics, i.e., actual motion in physical space, while Eq. (3) can be associated with mental dynamics describing information flows in the probability space. Such an interpretation [3] was evoked by the concept of reflection in psychology. Reflection is traditionally understood as the human ability to take the position of an observer in relation to one's own thoughts [11]. In other words, reflection is a self-awareness via the interaction with the "image of the self." In terms of the phenomenological formalism proposed above, Eq. (3) represents the probabilistic "image" of the dynamical system (2). If this system "possesses" its own image, then it can predict, for instance, future expected values of its parameters, and by interacting with the image, change expectations if they are not consistent with the objective. In this context, Eq. (1) simulates acting, and Eq. (2) simulates "thinking." Their interaction can be implemented by incorporating probabilities, their functions, and functionals into the control parameter μ [see Eq. (5)]. From the cognitive viewpoint, μ implements the self-awareness associated with the amount of information which the system possesses about its self-image.

In general, Eq. (3) is representable in the form (1), and therefore it possesses a variety of different complex patterns outside of equilibrium. However, in contrast to Eq. (1), Eq. (3) simulate patterns in probability space, i.e., in the space of the mental dynamics, so that the corresponding actual motions in physical space are described by nonlinear random walks (2). Because of this a species is not locked up in a certain pattern of behavior: it still can perform a variety of motions, and only the statistics of these motions is constrained by this pattern.

3. EMERGING SELF-ORGANIZATION.

We will start the analysis of the coupled motor–mental dynamics with Eqs. (2) and (3), where

$$p = \sin^2 (af_{t,x} + \beta), \qquad \mu = p - \frac{1}{2}, \qquad a, \beta = \text{const}, \qquad f = f(x, t) \tag{6}$$

i.e.,

$$x_{t+\tau} = x_t + h \operatorname{sgn}\left[R + \sin^2(af_{t,x} + \beta) + \frac{1}{2}\right] \tag{7}$$

$$f_{t+\tau,x} = f_{t,x-h} \sin^2(af_{t,x} + \beta) + f_{t,x+h} \cos^2(af_{t,x} + \beta) \tag{8}$$

Here a and β are constant weights, or control parameters.

In order to illustrate the fundamental nonlinear effects, we will analyze the behavior of special critical points by assuming that

$$a = \frac{5\pi}{12}, \qquad \beta = -\frac{\pi}{6} \tag{9}$$

$$f_0 = f(t = 0) = \begin{cases} f_0^{(1)} = \dfrac{1}{5} & \text{at} \quad x = -l \\ f_0^{(2)} = \dfrac{4}{5} & \text{at} \quad x = l \\ f_0^{(3)} = 0 & \text{otherwise} \end{cases}$$

Then the solution to Eq. (8) will consist of two waves starting from the points $x = -l$ and $x = l$, traveling toward each other with the constant speed $\upsilon = h/\tau$, and carrying the values $f_0^{(1)}$ and $f_0^{(2)}$, respectively, i.e.,

$$f = f_0^{(1)}\left(-l + \frac{h}{\tau} n\right) + f_0^{(2)}\left(l - \frac{h}{\tau} n\right), \qquad n = 0, 1, \ldots, \frac{l}{h} \tag{10}$$

where n is the number of time steps.

At $n = 1/h$, the waves undergo confluence into one solitary wave at $x = 0$:

$$f = \begin{cases} 1 & \text{at} \quad x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{at} \quad t = n\tau = \frac{l}{h}\tau \tag{11}$$

This process represents a discrete version of formation and confluence of shock waves, and it is characterized by a decrease of the Shannon entropy from

$$H(0) = -\frac{1}{5}\log_2\frac{1}{5} - \frac{4}{5}\log_2\frac{4}{5} > 0 \qquad \text{to} \qquad H(n\tau) = 0 \tag{12}$$

However, the solitary wave (11) is not stationary. Indeed, as follows from Eq. (8), the solution at $t = (n + 1)\tau$ splits into two values:

$$f_{(n+1)\tau} = \begin{cases} 1/2 & \text{at} \quad x = l \pm h \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

The process (13) can be identified as a discrete version of diffusion during which the entropy increases again from

$$H(n\tau) = 0 \qquad \text{to} \qquad H[(n + 1)\tau] = -\log_2\frac{1}{2} = 1 \tag{14}$$

The further evolutionary steps $t \geq (n + 2)\tau$ will include both diffusion and wave effects, and therefore the solution will endlessly display more and more sophisticated patterns of behavior in the probability space. The corresponding solutions to Eq. (7) represent samples of the stochastic process (8) in the form of nonlinear random walks in actual physical space.

Thus, the solutions to coupled motor–mental dynamics simulate emerging self-organization, which can start spontaneously. At this level of description, such an effect is triggered by instability rather than by a specific objective. In other words, the model represents a “brainless” life. However, it serves well the global objective of each living system: survival. Indeed, it is well established in biology [6] that marginal instability makes the behavior of a living system more flexible and therefore more adaptable to a changing environment.

The model (2), (3) is easily generalizable to three-dimensional motions:

$$x^{(i)}_{t+\tau} = x^{(i)}_{t} + h_i \operatorname{sgn}[(R + \mu_i)], \qquad i = 1, 2, 3 \tag{15}$$

$$f_{t+\tau, x^{(i)}} = \prod_{i=1}^{3} [p_i f_{t, x^{(i)}-h} + (1 - p_i) f_{t, x^{(i)}+h_i}] \tag{16}$$

Here $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are the space coordinates, and $f = f(x^{(1)}, x^{(2)}, x^{(3)}, t) = f^{(1)}f^{(2)}\mathrm{f}^{(3)}$ is the joint probability that the species occupies the point $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ at the instant t.

As in the one-dimensional case, here

$$p_i = \frac{1}{2} + \mu_i, \qquad 1, 2, 3$$

In particular, one can assume that

$$p_i = \sin^2(a_i f_{t,x} + \beta), \qquad i = 1, 2, 3 \tag{17}$$

It should be noticed that the nonlinear random walks (15) in all three directions are coupled by means of the joint probability f via the control parameters μ_i.

From the mathematical viewpoint, the model of mental dynamics (8) is linked to the Burgers equation in the sense that its pattern formation outside of equlibrium is based upon the balance between dissipation and shock waves.

In general, this model can be enriched with Belousov–Zhabotinskii effects by a slight modification of the random walk (7):

$$x_{t+\tau} = x_t + \frac{1}{2} h\{1 - \operatorname{sgn}[R + \mu]\} \operatorname{sgn}[R + \mu] \tag{18}$$

which now includes a third choice for the species: to remain at rest with the probability

$$q = \frac{1}{2} + \mu \tag{19}$$

The corresponding version of Eq. (8) reads

$$f_{t+\tau,x} = q f_{t,x} + p f_{t,x-h} + (1 - q - p) f_{t,x+h}, \qquad 0 \leq p, q \leq 1 \tag{20}$$

If $q = q(f)$ and $p = p(f)$, one arrives at the discretized version of the combined Burgers–Belousov–Zhabotinskii equation, which possesses a variety of new complex patterns outside of equilibrium that increase the adaptability of species to environmental changes. The Belousov–Zhabotinskii equation has been exploited for studying pattern formation in biology [1]. However, these patterns dwell in physical space; in contrast, Eqs. (18) and (20) simulate patterns in probability space, i.e., in the space of the mental dynamics, so that the corresponding actual motions in physical space are described by nonlinear random walks (7) and (18), respectively. Again, because of this, a species is not locked up in a certain pattern of behavior: it still can perform a variety of motions, and only the statistics of these motions is constrained by this pattern. It should be emphasized that such a "twist" is based upon the concept of reflection, i.e., the existence of a self-image.

4. FEEDBACK FROM EXPECTED FUTURE

The feedback (6) from mental to motor dynamics was expressed via the current probability distribution $f = f(x, t)$. In general, one can include in (6) memories $f = f(x, t - \tau)$ and nonlocal effects $f = f(x \pm h, t)$. In all these cases, the mental dynamics evolves independently of the motor dynamics. This property allows living systems to predict the future by running a self-image faster than real time, and then correct (if necessary) the motor dynamics (7) via the feedback from the expected future. Actually such a privilege of living systems represents the basic component of the concept of intelligence.

Let us now show how this phenomenon can be implemented in the model of motor–mental dynamics. For this purpose one has to modify the feedback (6) as follows:

$$p' = \sin^2 (\alpha' f_{t_0+\tau,x} + \beta'), \qquad \mu' = p' - \frac{1}{2}, \quad \alpha, \beta = \text{const} \tag{21}$$

where $f_{t_0+\tau}$ is found from Eq. (8).

It should be emphasized that $f_{t_0+\tau}$ is the expected distribution for $t > t_0$ since it is not yet affected by the new feedback (21). The real future distribution $f'_{t_0+\tau}$ is found from the modified mental dynamics:

$$f'_{t_0+\tau,x} = f'_{t,x-h}\, p' + f'_{t,x+h}\,(1 - p'), \qquad t > t_0 \tag{22}$$

in which p' is defined by Eq. (21). Equation (22) can be rewritten in explicit form for the case when, starting from $t_0 + \tau$, the feedback (6) is replaced by the feedback (21):

$$\begin{aligned} f'_{t_0+\tau,x} = {} & f'_{t,x-h} \sin^2\{\alpha'[f'_{t,x-h} \sin^2(\alpha f'_{t,x} + \beta) + f'_{t,x+h} \cos^2(\alpha f'_{t,x} + \beta)] + \beta'\} \\ & + f'_{t,x+h} \cos^2\{\alpha[f'_{t,x-h} \sin^2(\alpha f'_{t,x} + \beta) \\ & + f'_{t,x+h} \cos^2 \alpha f'_{t,x} + \beta)] + \beta\} \end{aligned} \tag{23}$$

The process described by Eq. (23) is still Markovian despite the fact that here the present is correlated with the future: indeed, as demonstrated above, the (expected) future is uniquely defined by the present [see Eq. (8)]. However, the process (23) is more sophisticated than those described by Eq. (8) and it can be used for systems with objectives since then the feedback from the expected future plays the role of a gradient (in the probabilistic space) which guides the evolution of the motor dynamics.

The feedback (21) can be presented in a more general form:

$$p' = \sin^2 F(f_{t_0+n\tau}), \qquad n = 1, 2 \ldots, \text{etc.}, \qquad \mu = p' - \frac{1}{2} \tag{24}$$

where F is an arbitrary function, and f_{t_0+nt} is the expected distribution at $t = t_0 + n\tau$ found from the equation

$$f_{t+\tau,x} = p_0 f_{t,x-h} + (1 - p_0) f_{t,x+h_j} \qquad p_0 = p' \quad \text{at} \quad t = t_0 \tag{25}$$

5. SYSTEMS WITH GLOBAL OBJECTIVES

As shown in the previous sections, the solutions to Eq. (8) or to its generalized versions, Eqs. (15), (22), and (25), can simulate the emergance of temporary self-organization, which is characterized by a decrease of the Shannon entropy [see Eq. (12)]. In this section we discuss a link between this phenomenon and global objectives of a living system.

First, as follows from the relation between finite differences E and derivatives D,

$$E_h = e^{hD} \tag{26}$$

the continuous representation of Eqs. (8), (15), (22), and (25) contains derivatives of all orders:

$$A_1 \frac{\partial f}{\partial t} + A_2 \frac{\partial^2 f}{\partial t^2} + \ldots = \beta_1 f + \beta_2 \frac{\partial f}{\partial x} + \beta_3 \frac{\partial^2 f}{\partial x^2} + \beta_4 \frac{\partial^3 f}{\partial x^3} + \ldots \quad (27)$$

while all the coefficients depend upon f and the control parameters α, β, α', β', etc. [see Eqs. (6), (17), (21), and (24)].

Therefore, the solutions to Eq. (27) may include the effects of the Burgers equation [7] ($A_1 = 1$, $B_2 = f$, $B_3 =$ const; the remaining A_1, $B_1 = 0$), the Korteweg–deVries equation [8] ($A_1 = 1$, $B_2 = f$, $B_4 = 1$; the remaining $A_1 B_i = 0$), and the Belousov–Zhabotinskii equation [1] ($A_1 = 1$, $B_1 \neq 0$, $B_3 =$ const; the remaining A_i, $B_i = 0$), which are, respectively, formation and diffusion of shock waves, formation of trains of solutions, and formation of trigger waves. The transitions from one pattern to another formally can be achieved by an appropriate change of the control parameters α, β, etc. In order to illustrate this, let us turn to the matrix representation of Eq. (3). We have

$$\pi(t + \tau) = \pi(t)P \quad (28)$$

Here the vector $\pi = \pi_1, \pi_2, \ldots, \pi_N$ represents the probability distribution $f(x, t)$ at the points $x = 1, 2, \ldots, N$, so that

$$\pi_i(t) = f(x_t, t), \qquad i = 1, 2, \ldots N, \qquad \sum_{i=1}^{N} \pi_i = 1 \quad (29)$$

For reflective boundary conditions at $x = 1$ and $x = N$, the matrix P has the following form [5]:

$$P = \begin{pmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1-p & p & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-p & 0 & p & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & p & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 1-p & p & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \quad (30)$$

If

$$p = \text{const} \quad (31)$$

Eq. (25) has a closed-form solution [5] which tends to a stationary distribution for large number of steps

$$n \gg N \quad (32)$$

and we have

$$\pi_k(n \gg N) \to \frac{p/(1-p) - 1}{(p/(1-p))^N - 1}\left(\frac{p}{1-p}\right)^{k-1} \tag{33}$$

and

$$p^n = \begin{pmatrix} \pi_1 & \cdots & \pi_N \\ \pi_1 & \cdots & \pi_N \\ \vdots & & \vdots \\ \pi_1 & & \pi_N \end{pmatrix} \qquad \text{for} \quad n \gg N \tag{34}$$

We pose the following problem: store a set of m stationary stochastic processes given by vectors of their probability invariants

$$I^{(i)} = I_1^{(i)}, I_2^{(i)}, \ldots, I^{(i)}; \qquad i = 1, 2, \ldots, m \tag{35}$$

(these invariants can be represented, for instance, by expectation E, variance, and higher moments) in such a way that when presented with any of the processes $\dot{\pi}^{(j)}$ out of the set of M processes

$$\dot{\pi}^{(j)} = \dot{\pi}_1^{(j)}, \dot{\pi}_2^{(j)}, \ldots, \dot{\pi}_N^{(j)}, \qquad j = 1, 2, \ldots, M \tag{36}$$

Eq. (28) converges to one of the stochastic processes (35).

The performance

$$\dot{\pi} \to I \tag{37}$$

represents the correspondence between two classes of patterns, i.e., a heteroassociative memory on a high level of abstraction. Indeed, each process in (37) stores an infinite number of different patterns of behavior, which, however, are characterized by the same sequence of invariants (35) and (36), representing a decision-making strategy. Considering a living system as a decision-maker, one can give the following interpretation of the mapping (37).

Classical artificial intelligence as well as artificial neural networks are effective in a deterministic and repetitive world, but faced with uncertainties and unpredictabilities, both of them fail. At the same time, many natural and social phenomena exhibit some degree of regularity only on a higher level of abstraction, i.e., in terms of some invariants. For instance, each particular realization of a stochastic process can be unpredictable in detail, but the whole ensemble of these realizations, i.e., “the big picture,” preserves the probability invariants (expectation, moments, information, etc.) and is therefore predictable in terms of behavior “in general.”

Hence, if the strategy of a decision-maker is characterized by a pattern $\dot{\pi}^{(i)}$ from (36), and starting from $t = 0$ external information becomes unavailable, it should change its strategy from pattern $\dot{\pi}^{(i)}$ to a corresponding pattern from (35), which can be associated with a decision based upon “common sense.”

It is implied that the attracting strategies I are sufficiently "safe," i.e., they minimize the risk taken by the decision-maker in case of an uncertain external world.

We will illustrate the approach by the simplest case when there are m attracting stochastic processes stored and each of them is characterized by only one invariant, the expectation $E^{(i)}$, $i = 1,2, \ldots, m$, while the rest of the invariant are not specified.

The first step in the implementation of the mapping (37) is to find an appropriate feedback. Departing from Eqs. (21) and (24), we will seek the nonlinear feedback in the form

$$p = \sin^2\left(\sum_{i=2}^{N-2} \alpha_i \pi_i + \alpha_1\right), \qquad \sum_{i=1}^{N} \pi_i = 1 \tag{38}$$

$$\pi_i = 0, \qquad \pi_N = 0 \tag{39}$$

Equation (38) introduces $N - 2$ weights α_i ($i = 1, 2, \ldots, N - 2$) to implement the mapping (37), while Eq. (39) enforces reflecting boundaries.

Substituting Eq. (38) into Eq. (33), one obtains

$$\pi_k = \frac{\tan^2\left(\sum_{i=2}^{N-2} \alpha_j \dot{\pi}_i + \alpha_1\right) - 1}{\tan^{2N}\left(\sum_{i=2}^{N-2} \alpha_i \dot{\pi}_i + \alpha_1\right) - 1} \tan^{k-1}\left(\sum_{i=2}^{N-2} \alpha_i \dot{\pi}_i + \alpha_1\right),$$

$$k = 1, 2, \ldots, N \ll n \tag{40}$$

One should notice that Eq. (33) (which was derived under the assumption $p =$ const) is still valid for the case (38) since starting with $n \gg N$, the stochastic process is supposed to be stationary (provided by an appropriate choice of the weights α_i), and therefore $\pi_i =$ const in Eq. (38)

The existence of m stationary stochastic processes with expectations $E^{(i)}$ ($i = 1, 2, \ldots, m$) requires that $N - 2$ weight coefficients α_i satisfy the following $m = N - 2$ equations:

$$E^{(i)} = \sum_{K=1}^{N-3} \left[k \frac{\tan^2\left(\sum_{j=1}^{N} \alpha_j \dot{\pi}_j^{(i)} + \alpha_0\right) - 1}{\tan^{2n}\left(\sum_{j=1}^{N} \alpha_j \dot{\pi}_j^{(i)} + \alpha_0\right) - 1} \tan^{k-1}\left(\sum_{j=1}^{N} \alpha_j \dot{\pi}_j^{(i)} + \alpha_0\right) - 1 \right],$$

$$i = 1, 2, \ldots, m \tag{41}$$

Here $E^{(i)}$ are expectations of the stochastic processes (35) to be stored, which are given, and $\dot{\pi}_j^{(i)}$ are probabilities of the stochastic processes (36) characterizing the original states of the system (which are also given). In order to solve this system in a dynamical way, one can apply the simplest version of the backpropagation strategy by minimizing the “energy”:

$$E = \sum_{i=1}^{N-2} \left\{ E^{(i)} - \sum \left[k \frac{\tan^{2}\left(\sum_{j=1}^{N-3} \alpha_j \dot{\pi}_j^{(i)} + \alpha_1\right) - 1}{\tan^{2N}\left(\sum_{j=1}^{N-3} \alpha_j \dot{\pi}_j^{(i)} + \alpha_1\right) - 1} \right.\right.$$

$$\left.\left. \times \tan^{k-1}\left(\sum_{j=1}^{N-3} \alpha_j \dot{\pi}_j^{(i)} + \alpha_1\right) - 1 \right] \right\} \rightarrow \min \tag{42}$$

i.e., by using the following recurrence relationships:

$$\alpha_i^{(j+1)} = \alpha_i^{(j)} - \lambda_i^2 \frac{E_i - E_{i-1}}{\alpha_i^{(j)} - \alpha_i^{(j-1)}}, \qquad \lambda_i^2 = \text{const}, \quad i = 1, 2, \ldots N - 2 \tag{43}$$

where

$$E_j = E(\alpha^{(j)}) \tag{44}$$

Here $\alpha^{(j)}$ and E_j are the jth approximations of the corresponding values of α and E.

It should be recalled that Eq. (43) represents a gradient dynamical system, and therefore the recurrence procedure in (43) always converges. However, since the nonlinearity of Eq. (42) (with respect to α_j) is not quadratic, the solution to Eq. (43) can be trapped in a local minima. Therefore, the recurrence procedure should be repeated several times starting from different initial values of $\alpha^{(0)}$, and then the lowest minimum of E has to be chosen as the solution.

Thus, the dynamical system (2), (3) represented in the equivalent form

$$x_{t+\tau} = x_t + h\, \text{sgn}(R + \mu) \tag{45}$$

$$\pi(t + \tau) = \pi(t)P \tag{46}$$

where the matrix P is expressed by Eq. (30), and

$$p = \sin^{(2)}\left(\sum_{j=2}^{N-3} \alpha_j \pi_j + \alpha_1\right), \qquad \alpha_j = \text{const}, \quad \pi_1 = 0, \quad \pi_N = 0, \quad \sum_{j=1}^{N} \pi_j = 1 \tag{47}$$

$$\mu = p - \frac{1}{2} \tag{48}$$

possesses the following property: If the initial value $x_{t=0}$ in Eq. (45) is drawn from a stochastic process $\dot{\pi}^{(j)}$ [see Eqs. (36)], then the solution to Eqs. (45) and (46) will approach (for $n \gg N$) a new stochastic process which is characterized by a prescribed expectation $E^{(j)}$ [see Eq. (32)]. This attracting stochastic process can be associated with the global objective of the underlying living system.

We will now make three comments concerning the mapping (37) introduced above.

First, strictly speaking, not all of the stochastic processes from Eq. (35) are true attractors: some of them can be repellers. Indeed, the corresponding weights α_j were found from static rather than dynamical conditions [see Eqs. (41)], and therefore the stability of these stochastic processes was not established. The situation here is similar to that in neural nets, where some of the equilibrium (or fixed) points are stable and some are not. In terms of Markov chains [see Eq. (46)], the possibility that some of the stochastic processes (35) are unstable (and therefore will never be approached) follows from the nonlinearity (47). It should be recalled that classical Markov chains are linear, and all the processes with reflecting boundaries coverge to stable stochastic processes.

Second, Eqs. (33), (34), (40), etc., include the condition that $n >> N$, i.e., that the number of time steps is significantly larger than the number of space steps N. Actually, this condition can be specified if one evokes a well-known result from the Markov chain theory [5] which quantifies Eq. (33):

$$\begin{aligned}\pi_{jk}^{(n)} = {} & \frac{p/(1-p)-1}{(p/(1-p)^{N}-1}\left(\frac{p}{1-p}\right)^{k-1} \\ & + \frac{2^{n+1}p^{1+1/2(n-j+k)}(1-p)^{1/2(n+j-k)}}{N}\sum_{r=1}^{N-1} S_{\Gamma}\ldots\end{aligned} \tag{49}$$

where

$$\begin{aligned}S_r = {} & \left\{\cos^{n}\frac{\pi r}{N}\left[\frac{\pi rj}{N}-\left(\frac{1-p}{p}\right)^{1/2}\sin\frac{\pi r(j-1)}{N}\right]\right. \\ & \left.\times\left[\sin\frac{\pi rk}{N}-\left(\frac{1-p}{p}\right)^{1/2}\sin\frac{\pi r(k-1)}{N}\right]\right\} \\ & \times\left\{1-2[p(1-p)]^{1/2}\cos\frac{\pi r}{N}\right\}^{-1}\end{aligned} \tag{50}$$

As $n \to \infty$, Eq. (49) tends to Eq. (33), while the second term in Eq. (49) allows one to evaluate n such that this term can be ignored in comparison to the first term.

Equation (50) [as well as Eq. (33)] was derived under assumption that $p = \text{const}$. However, it is still valid for the case (38) as long as the second term in Eq. (49) can be ignored since then the stochastic process is stationary, and therefore $\pi_i = \text{const}$ in Eq. (38). Obviously, this conclusion is true only if the stochastic process in (35) is stable. One should notice that $p = \text{const}$ is not necessarily the same as p at $t = 0$.

Third, if the attracting stochastic processes in Eq. (37) must be specified not only by the expectations (35), but also by higher moments, the additional [to Eq. (41)] constraints

$$\sum_{j=1}^{N-3} k\dot{\pi}_j^r = M_r^{(i)}, \qquad r = 2, 3, \ldots, r_x, \quad i = 1, 2, \ldots, m \tag{51}$$

should be imposed upon the weights α_j.

Here $M_r^{(r)}$ are the specified moments for the ith stochastic process, $\dot{\pi}_j$ are the probabilities expressed by Eq. (40), and m is the number of the attracting stochastic processes, and

$$m = \frac{N-2}{r} \tag{52}$$

The number of the prescribed moments m can be increased if the feedback (38) is generalized to the following form:

$$p = \sin^2\left(\alpha_1 + \sum_{i=2}^{N-2} \alpha_i \pi_i + \sum_{i,j=2}^{N-2} \alpha_{ij} \pi_i \pi_j + \ldots\right)$$

6. SYSTEMS WITH HIDDEN IDENTITY

A living system may have a global objective which is different from those described in the previous section. Indeed, let us consider a biological or social system in the form prey–predator game. Then it may be beneficial for a participant to make its behavior as unpredictable as possible. For a physical system, that would mean the maximization of the Shannon entropy subject to the constraints imposed upon its motion:

$$H = -\sum_{i=1}^{N} \pi_i \log \pi_i \to \max, \qquad \sum_{i=1}^{N} \pi_i = 1, \quad \text{etc.} \tag{53}$$

But a living system can do better than that: it can mislead its adversaries by hiding its identity.

The simplest way to do that is to make the feedback from mental to motor dynamics chaotic. For that purpose, rewrite Eq. (2) in the form

$$x_{t+\tau} = x_t + h\,\mathrm{sgn}\left[R + \frac{1}{2}\,\mu + \frac{1}{2}\left(\mu' - \frac{1}{2}\right)\right] \tag{54}$$

with the bias

$$\mu = \sin^2 \sum_{i=1}^{N} \alpha_i f_{t,x_i} - \frac{1}{2}, \qquad -\frac{1}{2} \le \mu \le \frac{1}{2} \tag{55}$$

$$\mu'_{t+\tau} = 4\mu_t(1 - \mu_t), \qquad 0 \le \mu' \le 1 \tag{56}$$

Equation (55) represents the deterministic components of the nonlinear feedback, and Eq. (56) is its chaotic component. Indeed, Eq. (56) is known as a logistic map, which leads to chaotic time series.

Now the transition probability in Eq. (3), with reference to Eq. (4), becomes

$$p_t = \frac{1}{2}\left(\sin^2 \sum_{i=1}^{N} \alpha_i f_{t,x_i} + \mu'_t + \frac{1}{2}\right), \qquad 0 \le p \le 1 \tag{57}$$

Since it includes the chaotic component $\frac{1}{2}\mu'$, the solution to Eq. (3) develops chaotic features, and therefore the probability distribution $f(t, x)$ becomes unpredictable.

Thus, the motor dynamics (54) is now run by the mental dynamics

$$f_{t+\tau,x} = p_t f_{t,x-h} + (1 - p_t) f_{t,x+h} \tag{58}$$

via the feedback (57), which includes a nonlinear deterministic component defined via the bias (55) and a chaotic component defined by the bias (56). The last component, which implements the hiding of identity by making the probability distribution $f(x, t)$ unpredictable, can be associated with a deception dynamics.

7. SYSTEMS WITH LOCAL OBJECTIVES

In many real-life situations, a living system does not know or cannot formulate a global objective. Instead, it can formulate local, i.e., a one-time-step-ahead objective. We will start with the simplest case: a predator–prey pursuit. We will assume that both the predator and prey possess not only the images of their selves, but images of their adversary as well. In terms of the three-dimensional model (15), (16), the pursuit can be formulated as follows:

$$x^{(i)}_{t+\tau} = x^{(i)}_{1} + h_i \operatorname{Sgn}(R + \mu^{(1)}_i), \qquad i = 1, 2, 3 \tag{59}$$

$$f^{(i)}_{t+\tau,x^i} = [p_i f^{(i)}_{t,x^{(i)}-h_i} + (1 - p_i) f^{(i)}_{t,x^{(i)}-h_i}], \qquad i = 1, 2, 3 \tag{60}$$

$$\tilde{f}^{(i)}_{t+\tau,y^{(i)}} = [\tilde{q}_i \tilde{f}^{(i)}_{t,y^{(i)}-h} + (1 - \tilde{q}_i) f^{(i)}_{t,y^{(i)}+h_i}], \qquad i = 1, 2, 3 \tag{61}$$

$$y^{(i)}_{t+\tau} = y^{(i)}_{t} + h_i \operatorname{sgn}(R + \mu^{(2)}_i), \qquad i = 1, 2, 3 \tag{62}$$

$$\varphi^{(i)}_{t+\tau,x^{(i)}} = [q_i \varphi^{(i)}_{t,y^{(i)}-h_\mathrm{i}} + (1 - q_i)\varphi^{(i)}_{t,y^{(i)}+h_i}], \qquad i = 1, 2, 3 \tag{63}$$

$$\tilde{\varphi}^{(i)}_{t+\tau,x^i} = \sum_{i=1}^{3} [\tilde{p}_i \tilde{\varphi}^{(i)}_{x^{(i)}-h_i} + (1 - \tilde{p}_i)\tilde{\varphi}^{(i)}_{x^{(i)}+h_i}], \qquad i = 1, 2, 3 \tag{64}$$

Here Eq. (59) simulates the motor dynamics of the predator, i.e., a random walk in three-dimensional space. Equation (62) describes the predator's mental dynamics, i.e., evolution of the probability $f(x^{(1)}, x^{(2)}, x^{(3)}, t) = f^{(1)}f^{(2)}f^{(3)}$, where the $x^{(1)}$ denote the predator's position, and p_1, p_2 and p_3 are the transition probabilities, which depend upon $f_{x_j} = f(t, x_j)$:

$$p_i = \frac{1}{2} + \mu^{(1)}_i = \sin^2 \sum_{j=1}^{N-1} (\alpha^{(i)}_j f^{(i)}_{x_j} + \alpha^{(i)}_j), \quad i = 1, 2, 3, \quad \sum_{j=1}^{N} f^{(i)}_{x_j} = 1 \tag{65}$$

where the $\alpha^{(1)}_j$ are constant weights to be found.

Equations (62) and (63) simulate the motor–mental dynamics of the prey, where the $y^{(i)}$ denote the prey's positions in space, and $\varphi(y^{(1)}, y^{(2)}, y^{(3)}, t) = \varphi^{(1)}\varphi^{(2)}\varphi^{(3)}$ is the corresponding probability: q_1, q_2, and q_3 are the transition probabilities

$$q_1 = \frac{1}{2} + \mu^{(2)}_i = \sin^2 \sum_{j=1}^{N-1} (\beta^{(i)}_j \varphi^{(i)}_{y_i} + \beta^{(i)}_j), \qquad i = 1, 2, 3, \quad \sum_{j=1}^{N} \varphi^{(i)}_{y_i} = 1 \tag{66}$$

Finally, Eqs. (61) and (64) simulate mental images of the adversaries: $\tilde{f}(\tilde{y}^{(1)}, \tilde{y}^{(2)}, \tilde{y}^{(3)}, t) = \tilde{f}^{(1)}\tilde{f}^{(2)}\tilde{f}^{(3)}$ and $\tilde{q}_i$ represent the prey's images in the "mind" of the predator, and $\tilde{\varphi}(\tilde{x}^{(1)}_1, \tilde{x}^{(2)}, \tilde{x}^{(3)}, t) = \tilde{\varphi}^{(1)}\tilde{\varphi}^{(2)}\tilde{\varphi}^{(3)}$ and $\tilde{p}_i$ represent the predator's images in the "mind" of the prey.

If the predator and the prey never met before, the best strategy for them is to assume that

$$\tilde{q}_i = (1 - p_i), \qquad \tilde{p}_i = (1 - q_i) \tag{67}$$

i.e., to consider the adversary as an extreme opposite to the self.

At this point, Eqs. (59), (60) and (62), (63) are coupled only in pairs, while Eqs. (61) and (64) are decoupled.

Now we introduce the objectives of the pursuit: the predator objective is to minimize the distance between the prey and itself during the next *n* steps, and the prey's objective is to maximize the same distance.

The distance after *n* steps is expressed as

$$E = \sum_{i=1}^{3} (x_k^{(i)} - y_k^{(i)})^2 \tag{68}$$

The only way to optimize this is to manipulate the weights α_i and β_i in Eqs. (65) and (66) using the strategy of the gradient descent approach [see Eqs. (42) and (43)]. However, here this strategy cannot be applied in a direct way since neither the predator nor the prey knows its actual future positions $x^{(i)}$ and $y^{(i)}$. Therefore, these positions have to be predicted based upon their images. The images can be represented by expectations, modes, or medians of the corresponding probability distributions. For instance, in case of expectations, the distance (68) is replaced by

$$\tilde{E} = \sum_{i=1}^{3} (\hat{x}_k^{(i)} - \tilde{y}_k^{(i)})^2 \tag{69}$$

where

$$\hat{x}_k^{(i)} = \sum x_k^{(i)} f^{(1)}, \qquad y_k^{(i)} = \sum y^{(i)} \varphi_k^{(i)} \tag{70}$$

Then the predator's and the prey's images of the same objective are, respectively,

$$\tilde{E}_1 = \left(\sum x^{(i)} f_k^{(i)} - \sum \tilde{y}^{(i)} \varphi_k^{(i)} \right)^2 \tag{71}$$

$$E_2 = \left(\sum \tilde{x}^{(i)} \tilde{f}^{(l)} - \sum y^{(i)} \varphi_k^{(i)} \right)^2 \tag{72}$$

These images are different since neither the predator nor the prey knows the actual probabilities *f* and φ of its adversary, and they replace them by the images $\tilde{\varphi}$ and $\tilde{f}$, respectively [see Eqs. (65)–(67)]. Now the strategy of the predator follows from the gradient descent minimization

$$a_{i(1)}^{(j+1)} = \alpha_{i(1)}^{(j)} - \lambda_1^2 \frac{\tilde{E}_1^{(k)} - \tilde{E}_1^{(j-1)}}{\alpha_{i(1)}^{j} - \alpha_{i(1)}^{(j-1)}}, \qquad \lambda_1 = \text{const} \tag{73}$$

$$\beta_{i(1)}^{(j+1)} = \beta_{i(1)}^{j} - \lambda_1^2 \frac{\tilde{E}_1^{(j)} - \tilde{E}_1^{(j-1)}}{\beta_{i(1)}^{j} - \beta_{i(1)}^{(j-1)}}, \qquad \lambda_1 = \text{const} \tag{74}$$

Similarly, the strategy of the prey follows from the gradient descent maximization

$$\alpha_i^{(j+1)} = \alpha_i^j + \lambda_2^2 \frac{\tilde{E}_2^{(j)} - \tilde{E}_2^{(j-1)}}{\alpha_i^j - \alpha_i^{(j-1)}}, \qquad \lambda_2 = \text{const} \tag{75}$$

$$\beta_i^{(j+1)} = \beta_i^{(j)} + \lambda_2^2 \frac{\tilde{E}_2^{(j)} - \tilde{E}_2^{(j-1)}}{\beta_i^{(j)} - \beta_i^{(j-1)}} \tag{76}$$

Thus, prior to each move, the predator and prey find the optimal weights α_i and β_i from Eqs. (73)–(76), plug them into Eqs. (59)–(64) via Eq. (65), and then make the next ("optimal") step.

There are four comments to be made concerning the model of pursuit. First, the system (59)–(64) is now fully interconnected via the objectives (71), (72) by means of Eqs. (73)–(76) and (65), (66). In particular, this means that the stochastic processes (60) and (63) are correlated. But this does not necessarily mean that there exists a joint probability function $\tilde{f}(\{x\},\{y\})$ for which $f^{(1)}$ *and* $f^{(2)}$ are the conditional probabilities. Indeed, as shown in refs. 8 and 9, the stochastic processes (60) and (61) are entangled in the sense that there is no such transformation of coordinates $\{x\}$, $\{y\}$ which would decouple them.

Second, each species, exploits the probabilistic images of the self and its adversary to predict future positions and to make the best available move, and this remarkable property, which is a privilege of living systems, can be associated with intelligence.

Third, success of the pursuit depends upon the degree of superiority of the predator's mental capacity over that of the prey if the mental capacity is measured by the speed of learning, i.e., by finding the correct values of the weights α and β from the gradient descent (73), (74).

Fourth, in the pursuit model, each species can demonstrate intelligence not only via the correct prediction of future moves, but also by making misleading moves based upon deception dynamics (56) if the feedbacks (65) and (66) are modified to the form (57).

Finally, as follows from the model, successful pursuit can be associated with the catching of a prey by a predator, which depends upon how well the predator predicts the prey's moves. The power to predict starts with Eq. (67) when the predator selects the transition probabilities for its image of the prey. The ability to make this simple and universal choice mimics the so-called innate properties conferred by all those elements with which the species was born. However, the choice (67), in general, may not be good enough: it does not include the specific characteristic of the prey. That is why the very first acquaintance between the adversaries may end in a failure on the part of the predator. But suppose that the predator managed to catch the prey. Then it can "record" the values of the actual objective (68) and its image (71). Therefore the difference

$$|\Delta E_1| = |E - \tilde{E}_1| \tag{77}$$

will serve as a measure of the mismatch between the image and reality. Based upon this difference, the predator can correct its image of the prey by appropriate change of $\tilde{q}_i$ in Eq. (67). Actually it should minimize the difference (77) as a function of $\tilde{\alpha}_i$ and $\tilde{\beta}_i$, where

$$\tilde{q}_i = q_i(\tilde{\alpha}_i, \tilde{\beta}_i), \qquad i = 1, 2, 3 \tag{78}$$

i.e.,

$$\tilde{\alpha}_i^{(j+1)} = \tilde{\alpha}_i^{(j)} - \lambda_1^2 \frac{|\Delta E_1^{(j)}| - |\Delta E_1^{(j-1)}|}{\tilde{\alpha}_i^{(j)} - \tilde{\alpha}_i^{(j-1)}}, \qquad \lambda = \text{const} \tag{79}$$

$$\tilde{\beta}_i^{(j+1)} = \tilde{\beta}_i^{(j)} - \lambda_1^2 \frac{|\Delta E_1^{(j)}| - |\Delta E_1^{(j-1)}|}{\tilde{\beta}_i^{(j)} - \tilde{\beta}_i^{(j-1)}} \tag{80}$$

It should be noted that the predator cannot rerun the actual trajectory after the pursuit has been ended: it can change $|\Delta E|$ only by rerunning the image of this trajectory, i.e., by simulating the solutions to Eqs. (59)–(61).

Eventually,

$$\lim_{j\to\infty} \tilde{\alpha}_i^{(j)} = \alpha_i, \qquad \lim_{j\to\infty} \tilde{\beta}_i^{(j)} = \beta_i \tag{81}$$

i.e., the predator's image of the prey coincides with the prey's self image.

Thus, if Eq. (67) is associated with innate properties, Eq. (80) can be associated with acquired or learned properties.

Obviously the prey learns from the same experience, and as a result, it can acquire some defenses by correcting its image of the predator, i.e., by departing from Eq. (67) to the following:

$$\tilde{p}_i = \hat{p}_i(\tilde{\alpha}_i', \tilde{\beta}_i'), \qquad i = 1, 2, 3 \tag{82}$$

where the weights $\tilde{\alpha}$ and $\tilde{\beta}$ are adjusted by means of minimizing the difference

$$|\Delta E_2| = |E - \tilde{E}_2| \tag{83}$$

8. SELF-REPRODUCTION

Self-reproduction is one of the privileges of living systems. In order to simulate it within the framework of our phenomenological formalism, we have to make the following assumption: all species of the same genotype have the same probabilistic invariants of their behaviors. In other words, their trajectories are different samples of the same stochastic process, i.e., they are different on the level of deterministic details, but are identical on the level of statistics. Then the self-reproduction process can be simulated by throwing into the "battlefield" newer and newer samples of the same stochastic process [for instance, the one described by Eq. (8)]. The rate of reproduction has to be governed by the logistic equation:

$$N_{j+1} = N_j - \gamma N_j(1 - N_j) \tag{84}$$

where N is the population density, and γ is the coefficient describing effects of food availability and death rate.

Thus, again we arrive at two different types of descriptions: the global picture is still expressed by Eq. (8), i.e., by the mental dynamics in the probability space; the local picture, or motor dynamics, is represented by a set of nonlinear random walks (7) whose density (in physical space) is expressed by Eq. (84).

If several genotypes occupy the same physical space, then the global picture (in the mental space) is represented by Eqs. (60), (61), (63), and (64), while the motor dynamics is described by two sets of random walks (59) and (62) whose densities are expressed by the corresponding versions of Eq. (84).

9. COLLECTIVE PERFORMANCE

In this section we will briefly describe collective phenomena in the proposed model, which combines the paradigms discussed above (on the level of individual or pairs of species) with the effects of swarms of species. In other words, we will depart from a single monad and move to a system of interacting (collaborating or competing) monads within the framework of dynamics of intelligent systems.

9.1. Collaboration

Suppose that there are several different, but "friendly" swarms of species having the probability distributions

$$f_j = f_j(\{x_j\}, \{y_j\}, \{z_j\}, t); \qquad j = 1, 2, \ldots, s \tag{85}$$

where $\{x_j\}$, $\{y_j\}$, and $\{z_j\}$ are sets of space coordinates occupied by the species of the jth swarm.

We will postulate that within the framework of our formalism, "friendliness" is equivalent to the existence of the joint probability

$$f = f(\{x_1\}, \ldots, \{z_s\}) \tag{86}$$

so that the f_j in Eq. (85) are interpreted as the conditional probabilities:

$$f_j = f_{\{j\}|\{\ \}}(\{x_1\}, \ldots, \{z_s\}) \tag{87}$$

As shown in ref. 8, that imposes upon f_j the following constraints:

$$\frac{\partial^2}{\partial x_j\,\partial x_k} \ln \frac{f_j}{f_k} = 0; \qquad j, k = 1, 2, \ldots, s; \quad j \neq k \tag{88}$$

If these constraints are satisfied, one can describe the joint evolution of all s swarms by only one equation:

$$f_{i+\tau,\{x\}} = \prod_{j=1}^{5} \prod_{i=1}^{3} [p_i^i f_{t\{x\}_i^i - \{h\}_i^i} + (1 - p_j^i) f_{t,\{x\}_j^i + \{h\}_j^i}] \tag{89}$$

which represents the evolution of the image of the whole set of species in the probability space. This evolution as a collective brain controls the motor dynamics of each species:

$$x_{t+\tau}^{(i)} = x_t^{(j)} + h_j \operatorname{sgn}(R + \mu_j), \qquad j = 1, \ldots, 3 \tag{90}$$

in a centralized way.

The simplest version of the dependence $p_j(f)$ is

$$p_j = \sin^2(\alpha_j f + \beta_j), \qquad \mu_j = p_j - \frac{1}{2} \tag{91}$$

which is similar to Eq. (6) for a single species. However, here the coefficients α_j and β_j may depend upon the population density N, i.e.,

$$\alpha_j = \alpha_j(N), \qquad \beta_j = \beta_j(N) \tag{92}$$

At the same time, the coefficient γ in Eq. (84) is likely to depend upon the probability f, i.e.,

$$\gamma = \gamma(f) \tag{93}$$

As a result, Eqs. (92) and (93) couple the motor–mental dynamics (90), (91) with the dynamics of the population density (84).

Hence, in addition to the multidimensional version of the nonlinear effects discussed for a single species, such as spontaneous self-organization, one can expect phenomena associated with a many-body problem: aggregation, formation of new alliances, explosions of population densities, etc.

9.2. Competition

Suppose the swarms described by the probabilities (85) are "hostile." In terms of our formalism this means that the constraints (88) are not satisfied, and therefore a joint probability (86) does not exist. In other words, the hostile swarms cannot be controlled by a unified "collective brain" as in the previous case. However, they can be entangled in a more sophisticated way. Indeed, here, instead of Eq (86), one arrives at a set of s coupled equations:

$$f^{(j)}_{t+\tau,\{x\}} = \prod_{i=1}^{3} [p_i^{(j)} f^{(j)}_{t,\{x\}^i_j - \{h\}^i_j} + (1 - p_i^{(j)}) f^{(j)}_{t,\{x\}^i_i + \{h\}^i_j}], \qquad j = 1, 2, \ldots, s \tag{94}$$

where

$$p_i^{(j)} = p_i^{(j)}(f^{(1)}, \ldots, f^{(s)}) \tag{95}$$

Each of these equations represents the evolution of the image of the corresponding swarm; however, these evolutions are coupled via Eqs. (95).

In order to emphasize the fundamental difference between the unified evolution (89), i.e., the collective brain, and the coupled evolutions (94), one has to recall that in physics the violation of compatibility conditions is usually associated with fundamentally new concepts or a new physical phenomenon. For instance, the incompatibility of velocities in a fluid, i.e., nonexistence of a velocity potential

$$\frac{\partial \upsilon_x}{\partial y} - \frac{\partial \upsilon_x}{\partial x} = \operatorname{curl} \upsilon \neq 0 \tag{96}$$

introduces vorticity and rotational flows. In the same way, the violation of Eqs. (88), i.e., nonexistence of a joint probability (89), leads to coupled evolution of the stochastic processes (94), while the degree of the incompatibility

$$\operatorname{ink}(f^{(j)}, f^{(k)}) = \frac{\partial^2}{\partial x_j\, \partial x_k} \ln \frac{f^{(j)}}{f^{(k)}} \tag{97}$$

can be interpreted as a some sort of “vorticity” in the probability space.

As mentioned earlier, the “vorticity” makes it impossible to find a transformation of the coordinates x_j that would decouple the stochastic processes (94), i.e., these processes are entangled.

Thus, the “vorticity” (97) brings a new dimension to the complexity of the motor–mental dynamics (89), (90): it makes the control of the motor dynamics of each species less centralized and more distributed. In addition, as shown in ref. 8, the information capacity of a set of entangled stochastic processes (94) is greater than that of the processes having the joint probability (89).

In the same way as described for a simple species, the evolutions (85) and (90) can be driven not only by nonlinear instability, but by objectives as well, which include learning, self–nonself discrimination (on the level of swarms), calibration, etc.

10. MINIMUM-FREE-WILL PRINCIPLE

In the previous discussion, to demonstrate the concept, the nonlinear function $p(f)$ was chosen in the simplest form (6) or (38). However, the only restriction imposed upon this function is the condition

$$0 \leq p \leq 1 \tag{98}$$

In general, it can be sought in the form

$$p = \sin^2[\varphi(f)] \tag{99}$$

where $\varphi(f)$ is an arbitrary function or a functional of f.

If this function is parametrized, for instance, as

$$\varphi(f, \alpha) = \sum_{k=0}^{q} \alpha_k f^k \tag{100}$$

then the weights α_k can be found from the objective by minimization of the corresponding functional [see Eqs. (41) or (73)–(76)].

But suppose that there are several different ways in which the same objective can be achieved, i.e., the function (100) includes a set of weights β

$$\varphi = \varphi(f, \alpha, \beta) \tag{101}$$

which do not affect the objective. How should the species solve such a redundancy problem?

Let us assume that the physical (i.e., the passive) component of the species motion is a symmetric random walk which is a discretized version of Brownian motion.

Then the transitional probability p in Eq. (99) can be decomposed as

$$p = \sin^2\left[\hat{\varphi}(f) + \frac{\pi}{4}\right], \qquad \text{i.e.,} \quad p = \frac{1}{2} \quad \text{if} \quad \hat{\varphi}(f) \equiv 0 \tag{102}$$

In this form, the nonlinear component of p, i.e., the function $\hat{\varphi}(f)$, represents the deviation from the passive motion, i.e., "free will."

Now, if a species can achieve its objective by several different ways, it will choose the one that minimizes the deviation from passive motion, i.e., it will minimize its free-will component. In other words, if a species is offered a "free ride" by physics, it should take it. This minimum-free-will principle can be associated with the Gauss minimum constraints principle, according to which the motion of a constrained system minimizes the deviation from the corresponding free motion. However, in contrast, the minimum-free-will principle is not required by physics, but it is imposed by biology. Indeed, a "crazy" species can move "against" the minimum-free-will principle, but it will waste its energy and "intellectual" effort, and as a result, its chances for survival will be decreased.

A natural measure of deviation from the passive motion is the difference

$$|\Delta H| = |H_0 - H| \tag{103}$$

where H_0 and H are the entropies of the passive and the actual motions, respectively.

Now we can give the mathematical formulation of the minimum-free-will principle: if the objective of the species is defined in the time interval $0 \leq t \leq T$, its motion will minimize the free-will measure

$$F_w = \sum_{t=0}^{T} |H_0 - H|_t \tag{104}$$

subject to the objective. In other words, if the weights α in Eq. (84) are defined by the objective, then the redundant weights β^* must be found from the condition

$$F_w(\beta^*) = \inf F(\beta) \tag{105}$$

Recall that

$$H = -\sum_{x=0}^{L} f_x \log_2 f_x, \qquad 0 \leq x \leq L$$

and f is found from Eq. (2), where $p = \sin^2[\varphi(f, \beta)]$.

11. REFLEXIVE CHAINS

As follows from the previous sections, the fundamental difference between a physical and an intelligent system is the possession of a self-image by the latter. In Section 7 we introduced a set of images: the self-image and the image of the adversary. In this section we will describe a general picture of the world of images, and in particular, the complexity of its reflexive structures.

The notion of reflexive structures was introduced in mathematical psychology [11] based upon an axiomatically defined formalism. In our approach, the same phenomenon is represented as an interaction between a stochastic process (describing motor dynamics, i.e., actual reality) and its own probability evolution (describing mental dynamics, i.e., self-image); no additional postulates are required. However, the feedback which controls motor dynamics includes a set of control parameters α_i [see, for example, Eq. (38)] which are supposed to be found from the objective [see Eqs. (42) and (43)]. Only when the same objective can be achieved by several different combinations of the control parameters, i.e., when these parameters are redundant, does the minimum-free-will principle have to be applied.

Let us consider a set of n interacting monads. Each monad is characterized by its motor coordinates $x_j(t)$ and self-image coordinates $f_j(t, x_j)$, $j = 1, 2, \ldots, n$. We associate such a state of the system with the first level of reflection. On the second level of reflection, each monad has images of the rest of the monads, i.e., $f_{jk}(t, x_k)$, $k \neq j$. On the third level of reflection, each monad has images of images of the rest of the monads of themselves as well as of others, i.e., $f_{jkl}(t, x_k, x_s)$, $k \neq j$, etc.

It can easily be verified that the number of images on the first level of reflection is

$$l_1 = n$$

and on the second level of reflection is

$$l_2 = n(n - 1) = n(l_1 - 1), \quad \text{etc.}$$

Finally,

$$\begin{aligned} l_m &= n(l_{m-1} - 1) \\ &= n + n(n - 1) + n[n(n - 1) - 1)] + \ldots \propto n^m \end{aligned} \tag{106}$$

where m is the level of reflection.

Thus, the number of images characterizing the state of an n-monad system grows polynomially as a function of n and exponentially as a function of m.

For an m level of reflection, the behavior of each monad is described by a system of $(l_m/n) + 1$ equations corresponding to the variables x_j, f_j, f_{jk}, f_{jkx}, f_{jksq}, etc. The first group of variables, x_j, characterizing the motor dynamics, is governed by equations of the type (59); the second group of variables, f_j, characterizing the dynamics of the self-image, is governed by equations of the type (60); the third group of variables, f_{jk}, characterizing the dynamics of the images of the other monads, is governed by equations of the type (61), etc. All these equations are coupled via the common objective of the type (68), which adds another set of dynamical equations of the type (73)–(76) governing the adjustments of the control parameters α_i, β_i, etc.

As follows from Eq. (106), the world of images is never complete: each new level of reflection brings in an additional set of images with the corresponding number of the governing dynamical equations, and that leads to deeper and deeper interactions between monads. It may happen that some monads have longer chains of images than others; obviously, in case of competition, these monads can better predict the evolution of the whole system, and that will give them advantage over those with shorter chains of images.

Let us assume now that the number of levels of reflection tends to infinity, i.e., $m \to \infty$, and, as follows from Eq. (106), the number of interacting images l_m as well as the number of the correspondings governing equations grow exponentially. Does the system's behavior tend to some limit pattern such, that starting with some larger m, it no longer changes? Even without rigorous mathematical analysis it is obvious that the answer to this question depends upon the structure of the objective. Indeed, suppose that the system consists of n monads, and let us start with the case when they are collaborating. Then their objective can be formulated in terms of minimization of some functional E of the coordinates $x_j(t)$, $f_j(t, x_j)$, etc., through the control parameters α_i under the assumption that this functional has a unique global minimum. Since each new level m of reflection brings in a new set of control parameters, obviously

$$E_{m+1} \leq E_m, \qquad \text{and} \qquad E_m \to \min \quad \text{at} \quad m \to \infty \tag{107}$$

This means that the system is attracted to a certain pattern of behavior when the number of reflection levels is sufficiently large, and therefore its complexity is limited.

If the functional E has several local minima, then each of them can be approached with some probability, i.e., the system may have several different limit patterns of behavior if $m \to \infty$.

Finally, for competing monads, the functional E may have only saddle points when a minimum with respect to one set of coordinates corresponds to a maximum with respect to another. In this situation, the system does not have any stable limit behavior, and it will endlessly increase its complexity as $m \to \infty$. But does such an unstable limit behavior have some stable invariants at $m \to \infty$ in the same way as chaos does? At this stage we do not have an answer to that question.

12. DISCUSSION AND CONCLUSION

There have been many attempts to describe the behavior of living systems by the mathematical formalism of classical physics, which includes Newtonian mechanics, thermodynamics, and statistical mechanics [1]. Notwithstanding the indisputable success of this approach, we will concentrate on its limitations. To illustrate, we start with the following example: consider a small physical particle in a state of random migration due to thermal energy, and compare its diffusion, or physical random walk, with a biological random walk performed by a bacterium which can be associated with the simplest biological particle, i.e., a monad. The fundamental difference between these two types of motions can be detected in probability space: the probability evolution of the physical particle (which can be associated with the Fokker–Planck equation) is always linear, and it has only one attractor—a stationary stochastic process where the motion is trapped. On the contrary, a typical probability evolution of a biological particle is nonlinear: it can have many different attractors, but eventually each attractor can be departed from without any "help" from outside. However, such a behavior violates the second law of thermodynamics unless there is another "hidden" object which interacts with the monad. In order to find this object, one has to turn to mathematical psychology, which postulates that a human possesses a self-image and interacts with it. In terms of the mathematical formalism of classical physics, a self-image can be represented by the probability evolution (mental dynamics) associated with the corresponding random walk (motor dynamics). Then the interaction between the monad and its self-image is implemented by the feedback from mental to motor dynamics, and that makes the probability evolution nonlinear. Actually, the deviation from linear evolution expresses the "free will" of the monad. From the physical viewpoint, the self-image is an external object, which reconciles the biological random walk with the second law of thermodynamics. (The need for a reconcilation was expressed by Schrödinger [13].)

Thus, it has been proposed that in order to capture dynamical invariants of the behavior of a living system, classical physics should be equipped with an additional postulate, namely, that each living system possesses a self-image. This self-image can be incorporated into the mathematical formalism of nonlinear dynamic which evolve in probability space. The only difference between classical and probabilistic nonlinear dynamics is in the additional constraints imposed upon the latter by the normalization conditions following from the definition of probability. Actually the self-image postulate can serve as a definition of living systems characterized by purposeful movements.

It has been demonstrated that within the formalism introduced above, a living system can predict the future in terms of probabilities due to the smoothness of evolution in probability space (such smoothness does not exist in actual space because of irregularities of a random walk). This ability, which increases chances for survival, can be considered as a basic component of intelligence.

The proposed model of the simplest biological particle, a monad, consists of a generator of stochastic processes which represents the motor dynamics in the form of nonlinear random walks, and a simulator of the nonlinear version of the Fokker–Planck equation which represents the mental dynamics. Both components can be implemented by physical hardware (neural networks, cellular automata, etc.), and thereby one can introduce artificial intelligence systems which have the same phenomenology as natural ones.

Coupled motor–mental dynamics can simulate such processes as emerging serf-organization, decision-making based upon "common sense," predator–prey evolutionary games, a collective brain, etc. Therefore, the proposed model can serve as a starting point for a unified approach to modeling the behavior of intelligent systems.

APPENDIX

The model of motor dynamics [Eqs. (2), (7), (15), etc.] has been presented in the form of a discrete automation since it was assumed that at any given moment the system stays in one of the few discrete states, while the transition between such states was neglected.

In this Appendix we describe a complete physical scenario which includes both discrete (probabilistic) and continuous (deterministic) components of the motor dynamics, thereby implementing Eqs. (2), (7), (15), etc., by "physical hardware," i.e., without random number generators. The approach is based upon non-Lipschitz dynamics [9, 10], which introduces discreteness and randomness in a natural way, i.e., without man-made devices.

Let us consider the dimensionless motion of a particle of unit mass driven by a non-Lipschitz force:

$$\dot{\upsilon} = \nu\upsilon^{1/3} \sin \omega t, \qquad \nu = \text{const}, \qquad [\upsilon] = \frac{m^{2/3}}{\sec^{5/3}} \tag{A1}$$

$$\dot{x} = \upsilon \tag{A2}$$

where υ and x are the particle velocity and position, respectively.

Subject to the zero initial condition

$$\upsilon = 0 \qquad \text{at} \qquad t = 0 \tag{A3}$$

Eq. (A1) has a singular solution

$$\upsilon = 0 \tag{A4}$$

and a regular solution

$$\upsilon = \pm\left(\frac{4\nu}{3\omega} \sin^2 \frac{\omega}{2}\, t\right)^{3/2} \tag{A5}$$

These two solutions coexist at $t = 0$, and this is possible because at this point the Lipschitz condition fails:

$$\left|\frac{\partial\dot{\upsilon}}{\partial\upsilon}\right|_{l\to 0} = \frac{1}{3}\, \nu\upsilon^{-2/3} \sin \omega t\Big|_{t\to 0} \to \infty \tag{A6}$$

Since

$$\frac{\partial \dot{\upsilon}}{\partial \upsilon} > 0 \qquad \text{at} \qquad |\upsilon| \neq 0, \qquad t > 0 \tag{A7}$$

the singular solution (A4) is unstable, and the particle departs from rest following the solution (A5). This solution has two (positive and negative) branches [since the power in (A5) includes the square root], and either branch can be chosen with probability p or $1 - p$, respectively. It should be noticed that as a result of (A5), the motion of the particle can be initiated by infinitesimal disturbances (such motion never can occur when the Lipschitz condition holds: an infinitesimal initial disturbance cannot become finite in finite time [8, 9]).

Strictly speaking, the solution (A5) is valid only in the time interval

$$0 \leq t \leq 2\pi/\omega \tag{A8}$$

and at $t \leq 2\pi/\omega$ it coincides with the singular solution (A4)

For $t > 2\pi/\omega$, Eq. (A4) becomes unstable, and the motion repeats itself to the accuracy of the sign in Eq. (A5).

Hence, the particle velocity υ performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (A5) alternate randomly after each period equal to $2\pi/\omega$.

Turning to Eq. (A2), one obtains the distance between two adjacent equilibrium positions of the particle:

$$\Delta x_i = x_i - x_{i-1} = \pm \int_0^{2\pi/\omega} \left(\frac{4v}{3\omega} \sin \frac{\omega}{2} t \right)^{3/2} dt = 64(3\omega)^{-5/2} v^{3/2} = \pm h \tag{A9}$$

Thus, the equilibrium positions of the particle are

$$x_0 = 0, \qquad x_1 = \pm h, \qquad x_2 = \pm h \pm h \ldots \tag{A10}$$

while the positive and negative signs randomly alternate with probabilities p and $1 - p$, respectively.

Obviously, the particle performs an unrestricted random walk: after each time period

$$\tau = 2\pi/\omega \tag{A11}$$

it changes its value on $\pm h$ [see Eq. (A10)].

The probability density $f(x, t)$ is governed by the following difference equation:

$$f(x, t + \tau) = pf(x - h, t) + (1 - p)f(x + h, t) \tag{A12}$$

which represents a discrete version of the Fokker–Planck equation, while

$$\int_{-\infty}^{\infty} f(x, t)\, dx = 1 \tag{A13}$$

We make several comments on the model (A1) and its solution.

First, the "viscous" force

$$F = -\nu\upsilon^{1/3} \tag{A14}$$

includes static friction [see Eq. (A6)], which actually causes failure of the Lipschitz condition. These type of forces are well known in theory of viscoplasticity [12]. It should be noticed that the power 1/3 can be replaced by any power of the type

$$k = \frac{2n - 1}{2n + 1}, \qquad n = 1, 2, \ldots \tag{A15}$$

with the same final result (A12). In particular, by selecting large n, one can make k close to 1, so that the force (A13) will be almost identical to its classical counterpart

$$F_c = -\nu\upsilon \tag{A16}$$

everywhere excluding a small neighborhood of the equilibrium point $\upsilon = 0$, while at this point

$$\frac{dF}{d\upsilon} \to \infty, \qquad \text{but} \qquad \left|\frac{\partial F_c}{\partial \upsilon}\right| \to 0 \qquad \text{at} \qquad \upsilon \to 0 \tag{A17}$$

Second, without the failure of the Lipschitz condition (A6), the solution to Eq. (A1) could not approach its equilibrium $\upsilon = 0$ in finite time, and therefore the paradigm leading to the random walk (A12) would not be possible.

Finally, we discuss the infinitesimal disturbances mentioned in connection with the instability of the solutions (A5) at $\upsilon = 0$. Actually, the original equation should be written in the form

$$\dot{\upsilon} = \nu\upsilon^{1/3} \sin \omega t + \varepsilon, \qquad \varepsilon \to 0 \tag{A18}$$

where ε represents infinitesimal disturbances. It should be emphasized that this process is not driving the solution of Eq. (A18): it only triggers the mechanism of instability which controls the energy supply via the harmonic oscillations sin ωt. As follows from Eq. (A18), the disturbance ε can be ignored when $\dot{\upsilon} = 0$ or when $\dot{\upsilon} \neq 0$, but the equation is stable, i.e., $\upsilon = \pi\omega$, $2\pi\omega, \ldots$. However, it becomes significant during the instants of instability when $\dot{\upsilon} = 0$ at $t = 0, \pi/2\omega$, etc. Indeed, at these instants, the solution to Eq. (A18) can be positive or negative if $\varepsilon = 0$ [see Eq. (A5)]. However, with $\varepsilon \neq 0$,

$$\operatorname{sign} \upsilon = \operatorname{sign} \varepsilon \qquad \text{at} \quad t = 0, \pi/2\omega, \ldots \tag{A19}$$

i.e., the sign of ε at the critical instances of time (A19) uniquely defines the evolution of the dynamical system (A18). Thus, the dynamical system (A18) creates a binary time series, which, in turn, generates a random walk paradigm [Eq. (A2)].

We will start with the case when

$$\varepsilon = \varepsilon_0 \sin \frac{1}{\upsilon}, \qquad \varepsilon_0 \to 0 \tag{A20}$$

The function sin(1/υ) oscillates about zero with unbounded frequency at $\upsilon \to 0$, and therefore, with equal probability it can be positive or negative at $\upsilon \to 0$ if the precision of its representation is finite.

Therefore, the statistical signature of the random walk described by Eqs. (A18), (A2), and (A20) is expressed by the solution to Eqs. (A12) and (A13) at $p = 1/2$. With the initial conditions

$$f(0, 0) = 1, \qquad f(x, 0) = 0 \qquad \text{if} \quad x \neq 0$$

it is a symmetric unrestricted random walk:

$$f(x, t) = C_n^m 2^{-n}; \qquad m = \frac{1}{2}(n + x); \qquad n = \text{integer}(2\omega t/\pi) \tag{A21}$$

Here the binomial coefficient should be interpreted as 0 whenever m is not an integer in the interval $[0, n]$, and n is the total number of steps.

One can verify (by substitution) that the function $\omega = \sin(1/\upsilon)$ is the solution to the following differential equation:

$$\frac{d\omega}{d\upsilon} + \frac{1}{\upsilon^2}\sqrt{1 - \omega^2} = 0 \qquad \text{or} \qquad \dot{\omega} = -\frac{\dot{\upsilon}}{\upsilon^2}\sqrt{1 - \omega^2}$$

$$\dot{\omega} = -\frac{\dot{\upsilon}}{\upsilon^2}\sqrt{1 - \omega^2} \tag{A22}$$

Both equations in (A22) suffer from a failure of the Lipschitz conditions at $\upsilon = 0$.

Thus, the probabilities described by Eq. (A12) are simulated by the dynamical system (A22) and (A2) without an explicit source of stochasticity (while the "hidden" source of stochasticity is in finite precision of the functions representation combined with the non-Lipschitz instability).

Combining several dynamical systems of the type (A22) and (A2) and applying an appropriate change of variables, one can simulate a probabilistic Turing machine which transfers one state to another with a prescribed transitional probability. Non-Markovian properties of such a machine can be incorporated by introducing time-delay terms in equation (A2).

$$\dot{x} = v(t) + \alpha_2 v(t - \tau_0) + \alpha_2 v(t - 2\tau_0) + \cdots. \tag{A23}$$

However, there is a more interesting way to enhance the dynamical complexity of the system (A22) and (A2). Indeed, let us turn to Eq. (A23) and introduce a feedback from Eq. (A2) to Eq. (A22) as follows.

$$\varepsilon = \varepsilon_0(\omega - x) \tag{A24}$$

Then the number of negative (positive) signs in the string (25) will prevail if $x > 0$ ($x < 0$) since the effective zero-crossing line moves down (up) away from the middle. Thus, when ($x = 0$) at $t = 0$, the system starts with an unrestricted random walk as described above, and $|x|$ grows. However, this growth changes signs in Eq. (A24) such that $\dot{x} < 0$ if $x > 0$, and $\dot{x} > 0$ if $x < 0$. As a result of that

$$x_{\max} \leq y_{\max} \qquad x_{\min} \geq y_{\min} \tag{A25}$$

where $y_{\max}$ and $y_{\min}$ are the largest and the smallest values in the time series $y(t)$, respectively. Hence, the dynamical system (A22), and (A2) simulates a restricted random walk with the boundaries (A25) implemented by the dynamical feedback (A24), while the probability

$$p(\text{sign } \varepsilon > 0) = \begin{cases} 0 & \text{if } x \geq y_{\max} \\ 1 & \text{if } x \leq y_{\min} \end{cases} \tag{A26}$$

For the sake of qualitative discussion, assume that p change linearly between $x = y_{\min}$ and $x = y_{\max}$, i.e.,

$$\mathrm{p} = \begin{cases} 0 & \text{if } x > y_{\max} \\ \dfrac{y_{\max} - x}{y_{\max} - y_{\min}} & \text{if } y_{\min} \leq x \leq y_{\max} \\ 1 & \text{if } x < y_{\min} \end{cases} \tag{A27}$$

Then the simulated restricted random walk is a solution to equations (A12) and (A27). An alternative approach to representation of the bias ε in Eq. (A18) was described in [9] where ε was sampled from the chaotic time series of the logistic map.

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